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ON THE EXISTENCE OF SOLUTIONS OF NONLINEAR INFINITE SYSTEMS OF PARABOLIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

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Abstract. We consider the Fourier first initial-boundary value problem for an infinite system of nonlinear differential-functional equations. The right-hand sides of the system are functionals of unknown functions of the Volterra type and this system being thus essentially coupled by this functional argument. The existence of the solutions to this problem is proved by the well-known Schauder fixed point theorem.

1. Introduction. We consider an infinite system of weakly coupled nonlinear differential-functional equations of parabolic type of the form

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z), \quad i \in S, \tag{1}$$

where

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i, \quad \mathcal{A}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\frac{\partial^2}{\partial x_j \partial x_k},$$

$x = (x_1,\ldots,x_m)$, $(t,x) \in (0,T] \times G := D$, $T < +\infty$, $G \subset R^m$, G is an open and bounded domain with the boundary $\partial G \in C^{2+\alpha} \cap C^{2-0} (0 < \alpha \le 1)$. S is an arbitrary set of indices (finite or infinite) and z stands for the mapping

$$z \ : \ S \times \overline{D} \ni (i,t,x) \to z^i(t,x) \in R,$$

composed of unknown functions z^i.

Let $B(S)$ be the Banach space of mappings

$$v : S \ni i \to v^i \in R,$$

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with the finite norm

$$\|v\|_{B(S)} := \sup\{|v^i| : i \in S\}.$$

Denote by $C_S(\overline{D})$ the Banach space of mappings

$$w : \overline{D} \ni (t,x) \to \Big(w(t,x) : S \ni i \to w^i(t,x) \in R\Big) \in B(S),$$

where the functions w^i are continuous in $\overline{D}$, with the finite norm

$$\|w\|_0 := \sup\{|w^i(t,x)| : (t,x) \in \overline{D}, i \in S\}.$$

A mapping $z \in C_S(\overline{D})$ will be called *regular* in $\overline{D}$ if the functions z^i $(i \in S)$ have continuous derivatives $\frac{\partial z^i}{\partial t}, \frac{\partial^2 z^i}{\partial x_j \partial x_k}$ in D for $j,k = 1,\dots,m$.

The case of finite systems $(B(S) = R^r)$ was treated in [**2**]. The case of infinite countable systems have been discussed in [**3**]–[**6**]and for an infinite countable S there is $B(S) = l^\infty$. In this paper S is an arbitrary infinite set of indices.

For system (1), we consider the following Fourier first initial-boundary value problem:

Find the *regular solution* (or briefly: *solution*) z of system (1) in $\overline{D}$ fulfilling the initial-boundary condition

$$z(t,x) = g(t,x) \text{ for } (t,x) \in \Gamma, \tag{2}$$

where $D_0 = \{(t,x) : t = 0,\ x \in \overline{G}\},\ \sigma = (0,T] \times \partial G,\ \Gamma = D_0 \cup \sigma$ and $\overline{D} = D \cup \Gamma$.

For $\tau,\ 0 < \tau \le T$, we denote $D^\tau = (0,\tau] \times G$, $\sigma^\tau = (0,\tau] \times \partial G$, $\Gamma^\tau = D_0 \cup \sigma^\tau$, $\overline{D}^\tau = D^\tau \cup \Gamma^\tau$. Obviously $D^T = D$.

In papers [**3**]–[**5**],have been used to solve the above problem monotone iterative methods. However, applying the monotone methods takes assuming the monotonicity of the right-hand side functions f^i with respect to the functional argument and the existence of a pair of a lower and an upper function for the considered problem (1),(2) in $\overline{D}$ (cp.[**2**]). These are not typical assumptions in existence and uniqueness theorems. In [**6**], the Banach fixed point theorem (contraction principle) has been used to prove the existence and uniqueness of the solutions to this problem. Now to prove the existence of the solution to this problem, we shall apply the Schauder fixed point theorem [**8**],[**10**]. Considering mainly Banach spaces of continuous and bounded functions, we give some natural sufficient conditions for the existence. We remark that the a priori estimates which appear while applying the Banach and the Schauder fixed point theorems are parallel to the above-mentioned assumptions in the theory of monotone iterative techniques. We notice that the case of the finite systems was studied by H. Ugowski [**9**].

2. Notations, assumptions and auxiliary lemmas. The Hölder space $C^{l+\alpha}(\overline{D}) := C^{(l+\alpha)/2,l+\alpha}(\overline{D})$, $(l = 0,1,2,\ldots; 0 < \alpha \le 1)$ is the space of continuous functions h in $\overline{D}$ whose all derivatives $\frac{\partial^{r+s}h}{\partial t^r \partial x^s} := D_t^r D_x^s h(t,x)$ $\quad (0 \le 2r+s \le l)$ exist and are Hölder continuous with exponent α $(0 < \alpha \le 1)$ in D, with the finite norm

$$|h|_{l+\alpha} := \sup_{\substack{P\in D\\ 0\le 2r+s\le l}} |D_t^r D_x^s h(P)| + \sup_{\substack{P,P'\in D\\ 0\le 2r+s\le l\\ P\neq P'}} \frac{|D_t^r D_x^s h(P) - D_t^r D_x^s h(P')|}{[d(P,P')]^\alpha},$$

where $d(P,P')$ is the parabolic distance of points $P = (t,x)$, $P' = (t',x') \in R^{m+1}$

$$d(P,P') = (|t-t'| + |x-x'|^2)^{\frac{1}{2}},$$

and $|x| = (\sum_{j=1}^m x_j^2)^{\frac{1}{2}}$.

By $C_S^{l+\alpha}(\overline{D})$ we denote the Banach space of mappings z such that $z^i \in C^{l+\alpha}(\overline{D})$ for all $i \in S$ with the finite norm

$$\|z\|_{l+\alpha} := \sup\left\{|z^i|_{l+\alpha} : i \in S\right\}.$$

The boundary norm $\|\cdot\|_{l+\alpha}^\Gamma$ of a function $\phi \in C_S^{l+\alpha}(\Gamma)$ is defined as

$$\|\phi\|_{l+\alpha}^\Gamma := \inf_\Phi \|\Phi\|_{l+\alpha},$$

where the greatest lower bound is taken over the set of all extensions Φ of ϕ onto $\overline{D}$.

Finally, by $|\cdot|_{l+\alpha}^{D^\tau}$ and $\|\cdot\|_{l+\alpha}^{D^\tau}$ we denote the suitable norms in the spaces $C^{l+\alpha}(\overline{D}^\tau)$ and $C_S^{l+\alpha}(\overline{D}^\tau)$, respectively.

We denote by $C^{k-0}(D)$ $(k = 1,2)$ the space of functions h for which the following norms are finite (see [**7**, p.190])

$$|h|_{1-0} := |h|_0 + \sup_{\substack{P,P'\in D\\ P\neq P'}} \frac{|h(t,x) - h(t',x')|}{|t-t'| + |x-x'|},\ |h|_{2-0} := |h|_{1-0} + \sum_{j=1}^m |D_{x_j} h|_{1-0}.$$

We assume that the operators $\mathcal{F}^i$ $(i \in S)$ are *uniformly parabolic* in $\overline{D}$ (the operators $\mathcal{A}^i$ are uniformly elliptic in $\overline{D}$), i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^m a_{jk}^i(t,x)\xi_j\xi_k \ge \mu \sum_{j=1}^m \xi_j^2$$

for all $\xi = (\xi_1,\ldots,\xi_m) \in R^m$, $(t,x) \in \overline{D}$, $i \in S$.

We assume that the functions

$$f^i : \overline{D} \times C_S(\overline{D}) \ni (t,x,s) \to f^i(t,x,s) \in R,\ i \in S,$$

are continuous and satisfy the following assumptions:

(H_f) they are *uniformly Hölder continuous* (with exponent α) with respect to t and x in $\overline{D}$, i.e., $f(\cdot,\cdot,s) \in C_S^{0+\alpha}(\overline{D})$;

(V) they satisfy the *Volterra condition*: for arbitrary $(t,x) \in \overline{D}$ and for arbitrary $s, \tilde{s} \in C_S(\overline{D})$ such that $s^j(\bar{t},x) = \tilde{s}^j(\bar{t},x)$ for $0 \le \bar{t} \le t, j \in S$, there is $f^i(t,x,s) = f^i(t,x,\tilde{s})$ $(i \in S)$.

(H_a) The coefficients $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$ $(j,k = 1,\dots,m,\ i \in S)$ in equations (1) are uniformly Hölder continuous (with exponent α) in $\overline{D}$, i.e., $a^i_{jk} = a^i_{jk}(\cdot,\cdot) \in C^{0+\alpha}(\overline{D})$ and a^i_{jk} belong to $C^{1-0}(\sigma)$.

From this there follows the existence of constants $K_1, K_2 > 0$ such that

$$\sum_{j,k=1}^{m} |a^i_{jk}|_{0+\alpha} \le K_1, \qquad \sum_{j,k=1}^{m} |a^i_{jk}|^{\Gamma}_{1-0} \le K_2, \quad i \in S.$$

(H_g) We assume that $g \in C_S^{2+\alpha}(\Gamma) \cap C_S^{1+\beta}(\Gamma)$, where $0 < \alpha < \beta < 1$.

Remark 1. We remark that if $g \in C_S^{2+\alpha}(\Gamma)$ and the boundary $\partial G \in C^{2+\alpha}$ then, without loss of generality, we can consider the homogeneous initial-boundary condition

$$z(t,x) = 0 \quad \text{for} \quad (t,x) \in \Gamma. \tag{3}$$

Accordingly, in what follows we confine ourselves to considering the homogeneous problem (1), (3) in $\overline{D}$ only. □

From the theorems on the existence and uniqueness of solutions of the Fourier first initial-boundary value problem for linear parabolic equations (see A.Friedman [**7**], Theorems 6 and 7, p.65 and Theorem 4, pp.191–201) we directly get the following lemmas.

Lemma 1. *Let us consider the linear initial-boundary value problem*

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x) & \text{in } D,\ i \in S, \\ \gamma(t,x) = g(t,x) & \text{on } \Gamma. \end{cases} \tag{4}$$

If $\delta \in C_S^{0+\alpha}(D)$, the assumptions ($H_a$), ($H_g$) hold and $\mathcal{F}^i[g^i](t,x) = \delta^i(t,x)$ on ∂G $(i \in S)$ then problem (4) has the unique solution γ and $\gamma \in C_S^{2+\alpha}(\overline{D})$.

Moreover, the following Schauder type $(2+\alpha)$ – estimate holds

$$\|\gamma\|_{2+\alpha} \le c\left(\|\delta\|_{0+\alpha} + \|g\|^{\Gamma}_{2+\alpha}\right), \tag{5}$$

where $c > 0$ is a constant depending only on the constants μ, K_1, α and the geometry of the domain D. □

Lemma 2. *We consider the linear homogeneous initial-boundary value problem*

$$\text{(5)}\qquad \begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x) & \text{in } D,\ i \in S, \\ \gamma(t,x) = 0 & \text{on } \Gamma. \end{cases}$$

Assume that $\delta \in C_S(\overline{D}), \partial G \in C^{2+\alpha} \cap C^{2-0}$ *and* (H_a) *hold. Let* $\delta(t,x)$ *vanish on* ∂G *and let* γ *be a solution of problem* (5). *Then, for any* β, $0 < \beta < 1$, *there exists a constant* $K > 0$, *depending only on* β, μ, K_1, K_2 *and the geometry of the domain* D, *such that the following a priori* $(1+\beta)$ – *estimate holds*

$$\text{(6)}\qquad \|\gamma\|_{1+\beta} \le K\|\delta\|_0.$$

Moreover, there exists a constant $\bar{K} > 0$ *depending on the same parameters as* K *such that*

$$\text{(7)}\qquad \|\gamma\|_{1+\beta}^{D^\tau} \le \bar{K}\tau^{\frac{1-\beta}{2}}\|\delta\|_0^{D^\tau}$$

for $0 < \tau \le T$. $\square$

Let $\eta = \eta(t,x) \in C_S(\overline{D})$. We define the nonlinear Nemytskiĭ operator $\mathbf{F}$

$$\mathbf{F} : \eta \to \mathbf{F}[\eta], \quad \mathbf{F} = \{\mathbf{F}^i : i \in S\},$$

setting

$$\mathbf{F}^i[\eta](t,x) := f^i(t,x,\eta),\ i \in S.$$

We assume that the operator $\mathbf{F}$ has the following properties, which hold for any τ, $0 < \tau \le T$:

(I) the operator $\mathbf{F}$ maps the space $C_S^{0+\alpha}(\overline{D}^\tau)$ into $C_S^{0+\alpha}(\overline{D}^\tau)$, and for each function $u \in C_S^{1+\alpha}(\overline{D}^\tau)$ satisfying $\|u\|_{1+\alpha}^{D^\tau} \le M$ the following estimate holds
$$\|\mathbf{F}[u]\|_0^{D^\tau} \le B(1 + \|u\|_1^{D^\tau}),$$
for some $B > 0$ independent of u;

(II) the operator $\mathbf{F}$ is continuous in the space $C_S^{1+\alpha}(\overline{D}^\tau)$ in the following sense: if $u_\nu, u \in C_S^{1+\alpha}(\overline{D}^\tau)$ and
$$\lim_{\nu\to\infty} \|u_\nu - u\|_{1+\alpha}^{D^\tau} = 0 \quad \text{then} \quad \lim_{\nu\to\infty} \|\mathbf{F}[u_\nu] - \mathbf{F}[u]\|_0^{D^\tau} = 0.$$

3. Theorem on the existence.

Theorem. *Let all the assumptions hold and* $\tau^* \in (0,T]$ *be a sufficiently small number. Then there exists a solution of the problem* (1), (3) *in the domain* $\overline{D}^\tau$, *where* $0 < \tau < \tau^* \le T$, *and* $z \in C_S^{2+\alpha}(\overline{D}^\tau) \cap C_S^{1+\beta}(\overline{D}^\tau)$, $0 < \alpha < \beta < 1$.

Proof of Theorem. Denote

$$A_M^{\tau,\alpha} = \{u : u \in C_S^{1+\alpha}(\overline{D}^\tau),\ \|u\|_{1+\alpha}^{D^\tau} \le M,\ u(t,x) = 0 \ \text{ on } \Gamma^\tau,$$
$$0 < \tau \le T, 0 < \alpha < 1\}$$

where $M > 0$ is a constant.

The set $A_M^{\tau,\alpha}$ is a closed convex set of $C_S^{1+\alpha}(\overline{D}^\tau)$.

For $u \in A_M^{\tau,\alpha}$ we define a mapping $\mathbf{T}$ setting

$$z = \mathbf{T}[u],$$

where z is a regular solution of the problem

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = \mathbf{F}^i[u](t,x) & \text{in } D^\tau,\ i \in S, \\ z(t,x) = 0 & \text{on } \Gamma^\tau. \end{cases} \tag{8}$$

From property (I) and Lemma 1, it follows that, for $u \in A_M^{\tau,\alpha}$, problem (8) has the unique solution $z \in C_S^{2+\alpha}(\overline{D}^\tau)$.

Moreover, by Lemma 2 and (7), for any positive $\theta,\ 0 < \theta < 1$, there exists a constant $\bar{K} = \bar{K}(\theta)$ that

$$\|z\|_{1+\theta}^{D^\tau} \le \bar{K}\tau^{\frac{1-\theta}{2}} \|\mathbf{F}[u]\|_0^{D^\tau}$$

for $0 < \tau \le T$ and by property (I), we obtain

$$\|z\|_{1+\theta}^{D^\tau} \le \bar{K}\tau^{\frac{1-\theta}{2}} B(1 + \|u\|_{1+\theta}^{D^\tau}).$$

If we assume that $\|u\|_{1+\alpha}^{D^\tau} \le M$ and

$$0 < \tau \le \min\{\Big[\frac{M}{\bar{K}B(1+M)}\Big]^{\frac{2}{\alpha-1}}, T\} := \tau^* \tag{9}$$

then for $\theta = \alpha$ we get finally

$$\|z\|_{1+\alpha}^{D^\tau} \le M.$$

Therefore, $\mathbf{T}$ maps the set $A_M^{\tau,\alpha}$ into itself, i.e., $\mathbf{T}(A_M^{\tau,\alpha}) := \{\mathbf{T}[u] : u \in A_M^{\tau,\alpha}\} \subset A_M^{\tau,\alpha}$ for $\tau,\ 0 < \tau \le \tau^*$.

Let $\theta = \beta$ and $0 < \alpha < \beta < 1$. Then from Lemma 2 it follows that the set $\mathbf{T}(A_M^{\tau,\alpha})$ is a bounded subset of the space $C_S^{1+\beta}(\overline{D}^\tau)$, therefore (see [**1**], Theorem 1.31, p.11 or [**7**], Theorem 1, p.188) this set is a precompact subset of $C_S^{1+\alpha}(\overline{D}^\tau)$.

To prove that the mapping $\mathbf{T}$ is continuous we notice that, if $u_\nu, u \in A_M^{\tau,\alpha}$ and $z_\nu = \mathbf{T}[u_\nu]$, $z = \mathbf{T}[u]$ then, by the definition of $\mathbf{T}$, we have

$$\begin{cases} \mathcal{F}^i[z_\nu^i - z^i](t,x) = \mathbf{F}^i[u_\nu](t,x) - \mathbf{F}^i[u](t,x) & \text{in } D^\tau,\ i \in S, \\ z_\nu(t,x) - z(t,x) = 0 & \text{on } \Gamma^\tau. \end{cases}$$

Using estimate (7) to this problem we obtain

$$\|\mathbf{T}[u_\nu] - \mathbf{T}[u]\|_{1+\beta}^{D^\tau} = \|z_\nu - z\|_{1+\beta}^{D^\tau} \le \bar{K}\tau^{\frac{1-\beta}{2}} \|\mathbf{F}[u_\nu] - \mathbf{F}[u]\|_0^{D^\tau}.$$

If we assume that

$$\lim_{\nu\to\infty} \|u_\nu - u\|_{1+\beta}^{D^\tau} = 0,$$

then, by property (II), we have

$$\lim_{\nu\to\infty} \|\mathbf{F}[u_\nu] - \mathbf{F}[u]\|_0^{D^\tau} = 0.$$

Finally by (6)

$$\lim_{\nu\to\infty} \|\mathbf{T}[u_\nu] - \mathbf{T}[u]\|_{1+\beta}^{D^\tau} = 0,$$

i.e., the mapping $\mathbf{T}$ is continuous.

Thus, finally, by the Schauder fixed point theorem ([**8**] or [**10**], Theorem 2.A, p.56) we conclude that the mapping $\mathbf{T}$ has a fixed point $z \in A_M^{\tau,\beta}$. Therefore z is a solution of problem (1), (3) and it belongs to $C_S^{1+\beta}(\overline{D}^\tau)$. By Lemma 1 it follows that z also belongs to $C_S^{2+\alpha}(\overline{D}^\tau)$, i.e., $z \in C_S^{2+\alpha}(\overline{D}^\tau) \cap C_S^{1+\beta}(\overline{D}^\tau)$, $0 < \alpha < \beta < 1$ for $0 < \tau \le \tau^*$, where τ^* defined by (9) is a sufficiently small number. □

REMARK 2. If we suppose additionally that (see [**7**], p.204):

(I') there exists a positive constant M_0 such that, for every $M > M_0$, we have

$$K\|\mathbf{F}[u]\|_0 \le M \text{ in } \overline{D}$$

for all functions $u \in C_S^{1+\alpha}(\overline{D})$ satisfying $\|u\|_{1+\alpha}^D \le M$, where K is the constant appearing in Lemma 2;

then problem (1), (3) has a solution in the whole domain $\overline{D}$. □

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